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# THEORY OF REPEATED SUPERPOSITION OF LARGE DEFORMATIONS: ELASTIC AND VISCOELASTIC BODIES

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**Abstract**—The basic constitutive relations and boundary-value problems of the theory of repeatedly superimposed large deformations of elastic and viscoelastic materials are presented. We then concentrate on the two-dimensional problems of nucleation and deformation of circular or elliptical holes in the finite strain formulation. The cases are examined when cavity shape is known either initially or in the final state, or (for a viscoelastic material) at some intermediate moment. The correction for nonlinear effects is found to of the order of 30% at the points of maximal stress concentration. 

## 1. INTRODUCTION

A need for consideration of problems on the superimposed large elastic deformations was explained by Novozhilov (1948). The simplest problem of this kind: plane strain superimposed on the homogeneous finite tension, was examined by Green and Adkins (1960). The first practical results relating to this field are probably the equilibrium equations written in coordinates counted off from the metric determined by the initially stressed state. These equations were derived by Bondar in 1962 and were presented by Sedov (1965). Washizu (1982) explained in detail an approach to repeatedly superimposed large deformations (incremental theories).

In the present work the basic constitutive relations and boundary-value problems are formulated for repeatedly superimposed large deformations of elastic and viscoelastic materials. The theory specifically permits one to solve problems in which boundaries and boundary conditions, including connectedness of the region occupied by a body are changed repeatedly in the process of loading. This theory permits one to solve nonlinear problems of the new type, for example, stress concentration problems where the new stress concentrators develop in the process of loading by removal of parts of previously loaded body. As an example, plane problems of stress concentration around a circular or elliptical cavity nucleated in a body undergoing large deformations are considered. In one special case, the hole assumes a given shape at the moment of formation; in another case it assumes the prescribed shape at the final state (in the case of a viscoelastic body it assumes the given shape at a given moment of time).

## 2. KINEMATIC RELATIONS

Here the basic kinematic relations are presented for the case of repeatedly superimposed large deformations.

Let us distinguish $N$ states of a body: initial (undeformed) state; $(N-2)$ intermediate states, for which body goes step by step by successively applied external effects or, in the case of the viscoelastic material, processes taking place in it; final or current state, to which a body goes after the application of all external loads to it in the predetermined order (in the case of viscoelastic body, every external load is applied at a given moment). By "application of load" we mean application of body forces and application or removal of load over both the pre-existing boundaries of regions and the newly formed ones.

Let $u_n(\xi^k, t)$ denote the displacement vector from the preceding state to the following one. Hereafter $\xi^k$ denotes the Lagrangian coordinates and $t$ denotes time.

Representing the position vector of a material particle in the "$n$"-th intermediate step in the form

$$\overset{n}{r} = \overset{m}{r} + \sum_{p=m+1}^{n} u_p, \tag{1}$$

so the base vectors in this step are:

$$\overset{n}{e}_i = \frac{\partial \overset{n}{r}}{\partial \xi^i} = \frac{\partial \overset{m}{r}}{\partial \xi^i} + \sum_{p=m+1}^{n} \frac{\partial u_p}{\partial \xi^i}, \tag{2}$$

and using base $\overset{n}{e}_i$ for differentiation of vectors $u_p = \overset{m}{u}{}^j_p \overset{m}{e}_j$ in (2) we obtain the relation between the base vectors in the "$n$"-th and "$m$"-th steps:

$$\overset{n}{e}_i = \left( \overset{m}{g}{}^j_i + \sum_{p=m+1}^{n} \overset{m}{\nabla}_i \overset{m}{u}{}^j_p \right) \overset{m}{e}_j, \quad (n > m), \tag{3}$$

where

$$\overset{m}{\nabla}_i = \frac{\partial}{\partial \xi^i} \overset{m}{e}{}^i$$

(no summation over $i$).

Hereafter superscript over any symbol (excluding $r, e$) indicates the number of intermediate state.

It follows from (3) that the base vectors for a certain state may be defined arbitrarily; then, the base vectors for other states are determined uniquely.

Relations between $\mathrm{d}\overset{0}{r}, \mathrm{d}\overset{1}{r}, \ldots \mathrm{d}\overset{N}{r}$ for the same element in different states are established with the use of deformation gradients:

$$\mathrm{d}\overset{p}{r} = \mathrm{d}\overset{q}{r} \cdot \Psi_{q,p}, \tag{4}$$

where

$$\Psi_{q,p} = \overset{q}{e}{}^i \overset{p}{e}_i, \tag{5}$$

and also

$$\Psi_{q,p} = \Psi_{q,f} \cdot \Psi_{f,p}; \quad \Psi_{q,p} = \Psi_{p,q}^{-1} \tag{6}$$

$$\Psi_{q,p} = I + \sum_{n=q+1}^{p} \overset{q}{\nabla} u_n = \left( I - \sum_{n=q+1}^{p} \overset{q}{\nabla} u_n \right)^{-1}; \quad \overset{q}{\nabla} = \frac{\partial}{\partial \xi^i} \overset{q}{e}{}^i; \tag{7}$$

$I$ is the identity tensor; $\Psi_{q,p}$ is the correspondent deformation gradient.

Following the usual definition of strain tensors we consider a half of difference of squares of one and the same particle in transition from the "$q$"-th state to the "$p$"-th one:

$$\frac{1}{2}\left(\mathrm{d}\overset{p}{r}{}^2-\mathrm{d}\overset{q}{r}{}^2\right)=\mathrm{d}\overset{0}{r}\cdot\overset{0}{E}_{q,p}\cdot\mathrm{d}\overset{0}{r}=\mathrm{d}\overset{1}{r}\cdot\overset{1}{E}_{q,p}\cdot\mathrm{d}\overset{1}{r}=\cdots=\mathrm{d}\overset{m}{r}\cdot\overset{m}{E}_{q,p}\cdot\mathrm{d}\overset{m}{r}=\cdots=\mathrm{d}\overset{N}{r}\cdot\overset{N}{E}_{q,p}\cdot\mathrm{d}\overset{N}{r},\tag{8}$$

where, as follows from (1)–(8),

$$\overset{m}{E}_{q,p}=\frac{1}{2}(\Psi_{m,p}\cdot\Psi^{T}_{m,p}-\Psi_{m,q}\cdot\Psi^{\mathrm{T}}_{m,q}).\tag{9}$$

Here, $\overset{m}{E}_{q,p}$ is the strain tensor describing change of strains in transition from the "$q$"-th state to the "$p$"-th state and defined in the coordinate basis of the "$m$"-th state. We refer to $E_{0,1}$ as an initial strain tensor, to $E_{m,n}$ $(1\leqslant m<n\leqslant N)$ as an additional strain tensor and $E_{o,N}$ as a total strain tensor, where $\overset{0}{E}_{0,1}$ is the Green strain tensor and $\overset{1}{E}_{0,1}$ is the Almansi strain tensor, see Eringen (1962), Leigh (1968), Marsden and Huges (1983).

It further follows from the above written equations that:

$$\overset{m}{E}_{q,p}=\Psi_{m,n}\cdot\overset{n}{E}_{q,p}\cdot\Psi^{\mathrm{T}}_{m,n},\tag{10}$$

$$\overset{m}{E}_{q,p}=\overset{m}{E}_{q,f}+\overset{m}{E}_{f,p},\tag{11}$$

so that the covariant components of strain tensors $E_{q,p}$, computed in different coordinate bases $\overset{p}{e}{}^{i}$, are the same.

Using (7)–(11), one can represent $E_{q,p}$ in terms of displacement gradients:

$$\overset{m}{E}_{q,p}=\frac{1}{2}\left[\sum_{n=q+1}^{p}\left(\overset{m}{\nabla}u_n+u_n\overset{m}{\nabla}\right)+\alpha-\beta\right],\tag{12}$$

where

$$\alpha=\begin{cases}\displaystyle\sum_{n=m+1}^{p}\sum_{j=n+1}^{p}\overset{m}{\nabla}u_n\cdot u_j\overset{m}{\nabla}, & \text{if } m<p\\ \displaystyle\sum_{n=p+1}^{m}\sum_{j=p+1}^{m}\overset{m}{\nabla}u_n\cdot u_j\overset{m}{\nabla}, & \text{if } m>p\end{cases}$$

$$\beta=\begin{cases}\displaystyle\sum_{n=q+1}^{m}\sum_{j=q+1}^{m}\overset{m}{\nabla}u_n\cdot u_j\overset{m}{\nabla}, & \text{if } m>q\\ \displaystyle\sum_{n=m+1}^{q}\sum_{j=m+1}^{q}\overset{m}{\nabla}u_n\cdot u_j\overset{m}{\nabla}, & \text{if } m<q.\end{cases}$$

As an example we present expressions for $\overset{2}{E}_{q,p}$ in the space of the second state under three-step loading of a body:

$$\overset{2}{E}_{0,1}=\frac{1}{2}\left(\overset{2}{\nabla}u_1+u_1\overset{2}{\nabla}-\overset{2}{\nabla}u_1\cdot u_1\overset{2}{\nabla}-\overset{2}{\nabla}u_2\cdot u_1\overset{2}{\nabla}-\overset{2}{\nabla}u_1\cdot u_2\overset{2}{\nabla}\right)\tag{13}$$

$$\overset{2}{E}_{1,2}=\frac{1}{2}\left(\overset{2}{\nabla}u_2+u_2\overset{2}{\nabla}-\overset{2}{\nabla}u_2\cdot u_2\overset{2}{\nabla}\right)$$

$$\overset{2}{E}_{0,2}=\overset{2}{E}_{0,1}+\overset{2}{E}_{1,2}\tag{14}$$

$$\overset{2}{E}_{2,3} = \frac{1}{2}\left(\overset{2}{\nabla} u_3 + u_3 \overset{2}{\nabla} + \overset{2}{\nabla} u_3 \cdot u_3 \overset{2}{\nabla}\right)$$

$$\overset{2}{E}_{1,3} = \overset{2}{E}_{1,2} + \overset{2}{E}_{2,3} \tag{15}$$

$$\overset{2}{E}_{0,3} = \overset{2}{E}_{0,2} + \overset{2}{E}_{2,3}. \tag{16}$$

We note that, as follows from (12), at all the cases with the exception of $m = p$, $m = q$ and $p = q+1$, tensors $\overset{m}{E}_{q,p}$ parametrically depend not only on $\overset{m}{\nabla} u_m$, but also on $\overset{m}{\nabla} u_n$ $(m < n)$. For example, as follows from (13), $\overset{2}{E}_{0,1}$ depends not only on $\overset{2}{\nabla} u_1$, but also on $\overset{2}{\nabla} u_2$. It causes additional difficulties when problems are formulated in coordinate bases of the "later" states, since tensors specifying accumulation of strains in the earlier states and defined in coordinate basis of the later state depend, in particular, on displacement gradients describing transition to the later state. This dependence significantly complicates solution of problems, formulated in terms of the "later" state. However, such formulation is unavoidable in the cases, when the boundary conditions are formulated (fully or partly) namely in this state.

## 3. CONSTITUTIVE EQUATIONS

Here we represent well known constitutive equations of elasticity and viscoelasticity in the spaces of different states.

The following notations are used here:

$\overset{n}{\sigma}_{0,n}$—"true" stress tensor for the "$n$"-th state (that is the tensor of stresses accumulated in the body in transition to the "$n$"-th state and computed per unit area of this state); at $n = 1$ $\overset{1}{\sigma}_{0,1}$ is the Cauchy stress tensor;

$$\overset{n}{\Sigma}_{0,n} = (1+\Delta_{0,n})\, \overset{n}{\sigma}_{0,n}. \tag{17}$$

$\Delta_{0,n}$ is the relative volume change of a particle in transition from the initial to the "$n$"-th state:

$$(1+\Delta_{0,n}) = \prod_{m=0}^{n-1} (1+\Delta_{m,m+1}). \tag{18}$$

As shown by Levin and Tarasiev (1980) and Levin (1988), consideration of elementary work of true stress for the isotropic bodies yields:

$$\overset{m}{\Sigma}_{0,n} = \Psi^{\mathrm{T}}_{n,m} \cdot \overset{n}{\Sigma}_{0,n} \cdot \Psi_{n,m}. \tag{19}$$

From (19) and (6) we have

$$\overset{p}{\Sigma}_{0,n} = \Psi^{\mathrm{T}-1}_{p,q} \cdot \overset{q}{\Sigma}_{0,n} \cdot \Psi^{-1}_{p,q}. \tag{20}$$

Equation (19) sets up a relation between the generalized total for the "$n$"-th state stress tensor $\overset{n}{\Sigma}_{0,n} = (1+\Delta_{0,n})\overset{n}{\sigma}_{0,n}$ (characterizing accumulation of stresses by a particle in transition from the initial to the "$n$"-th state) and the generalized total for the "$n$"-th state stress tensor defined in the coordinate basis of an arbitrary "$m$"-th state $\overset{m}{\Sigma}_{0,n} = \overset{m}{\Sigma}{}^{ij}_{0,n}\, \overset{m}{e}_i \overset{m}{e}_j$.

Equation (20) sets up a relation between the generalized total for the "$n$"-th state stress tensors defined in the coordinate bases of an arbitrary "$p$"-th and "$q$"-th states. For convenience, below we shall write "stress tensor $\overset{m}{\sum}_{0,n}$" instead of "generalized total for the '$n$'-th state stress tensor defined in the coordinate basis of the '$m$'-th state". Note that when deformations are not superimposed,

$$\overset{0}{\sum}_{0,1}\cdot\Psi_{0,1} = P,\quad \overset{1}{\sum}_{0,1} = T_{(0)},\quad (1+\Delta)^{-1}\overset{0}{\sum}_{0,1} = T^{\mathrm{V}},$$
$$\overset{0}{\sum}_{0,1} = T^{X},\quad (1+\Delta)^{-1}\overset{1}{\sum}_{0,1} = \overset{1}{\sigma}_{0,1} = T \tag{21}$$

where $P$ is the first Piola stress tensor; $T$ is the Cauchy true stress tensor; $T^{\mathrm{V}}$ is the energetic (reduced) stress tensor, used by Lur'e (1980); $T_{(0)}$ is the Hamel–Kappus–Trefftz stress tensor, used in partition by Hamel (1949); $T^{x}$ is the second Piola–Kirchhoff stress tensor.

It is clear from (20) that the contravariant components of the generalized stress tensors $\overset{m}{\sum}_{0,n} = \overset{m}{\sum}{}^{ij}_{0,n}\overset{m}{e}_i\overset{m}{e}_j$ are the same for different values of $m$.

Let, for example, the following relation be defined:

$$\overset{n}{\sum}_{0,n} = \overset{n}{a}_0\left(\overset{n}{E}_k\right)I+\overset{n}{a}_1\left(\overset{n}{E}_k\right)\overset{n}{E}_{0,n}+\overset{n}{a}_2\left(\overset{n}{E}_k\right)\left(\overset{n}{E}_{0,n}\right)^2 \tag{22}$$

where $\overset{n}{E}_k$ $(k = 1,2,3)$ are the invariants of tensor $\overset{n}{E}_{0,n}$; $\overset{n}{a}_i(\overset{n}{E}_k)$ are the functions of these invariants.

Let us derive the constitutive equations for the tensor $\overset{m}{\sum}_{0,n}$. To do this we multiply the eqn (22) by $\Psi^{\mathrm{T}}_{n,m}$ on the left and $\Psi_{n,m}$ on the right. Taking into account the identities (6), (9), (10), (20), we have:

$$\overset{m}{\sum}_{0,n} = \overset{m}{b}_0\left(I+2\overset{m}{E}_{m,n}\right)^{-1}+\overset{m}{b}_1\left(I+2\overset{m}{E}_{m,n}\right)^{-1}\cdot\overset{m}{E}_{0,n}\cdot\left(I+2\overset{m}{E}_{m,n}\right)^{-1}$$
$$+\overset{m}{b}_2\left[\left(I+2\overset{m}{E}_{m,n}\right)^{-1}\cdot\overset{m}{E}_{0,n}\right]^2\cdot\left(I+2\overset{m}{E}_{m,n}\right)^{-1} \tag{23}$$

where $\overset{m}{b}_i$ $(i = 0,1,2)$ are the functions of simultaneous invariants of tensors $\overset{m}{E}_{0,n}$ and $\overset{m}{E}_{m,n}$.

Let us call this procedure used to obtain the constitutive equations for $\overset{m}{\sum}_{0,n}$ the "non-energetic transition" procedure.

When the constitutive equations for $\overset{0}{\sum}_{0,n}$ are specified, the representation for $\overset{m}{\sum}_{0,n}$ is found similarly. It should be noted that there are no restrictions for the value of $m$, that is, both cases $m < n$ and $m > n$ are possible. This means that tensor $\overset{m}{\sum}_{0,n}$ depends not only on $\overset{m}{E}_{0,n}$, but also on additional strain tensors $\overset{m}{E}_{m,n}$. It causes, at $m > n$, additional difficulties when problems are formulated in coordinate basis of the "$m$"-th state as already have been noted in the second paragraph. Similar techniques can be used to obtain constitutive equations for the generalized stress tensors for bodies of viscoelastic material in the spaces of different states, when these equations are known for a certain state.

We will concentrate now on the special case when the material is incompressible and the constitutive equations are:

$$\overset{0}{\sum}_{0,n} = \bar{\mu}[I-\tfrac{1}{3}G_1G^{-1}_{0,n}]-p_{0,n}G^{-1}_{0,n} \tag{24}$$

where $G^{-1}_{0,n} = \Psi^{T-1}_{0,n}\cdot\Psi^{-1}_{0,n}$; at $n = 1$, $G_{0,1}$ is the Cauchy–Green strain tensor, and $\overset{0}{\sum}_{0,1}$ is the "energetic" stress tensor; $G_1 = G_{0,n}\cdot\cdot I$; $p_{0,n}$ is a scalar; $\bar{\mu}$ is the relaxation operator:

$$\bar{\mu}\varphi(t) = \mu_0\left[\varphi(t) - \int_0^t 1(t-\tau)\varphi(\tau)\,\mathrm{d}\tau\right]. \tag{25}$$

Relation (25) is obtained on the basis of experimental data for creep and relaxation kernels of polymeric and rubber-like materials (for instance, polydienepoxyurethane-based built-in material) and had been used for solution of specific problems. Note that this relation belongs to the class of constitutive relations for simple materials, see Truesdell (1972). Using the non-energetic transition procedure we have:

$$\overset{m}{\sum}_{0,n}(t) = \Psi_{0,m}^{\mathrm{T}}\cdot\bar{\mu}\left[I - \frac{1}{3}G_1\Psi_{0,m}^{T-1}\cdot\left(I+2\overset{m}{E}_{m,n}(t)\right)^{-1}\cdot\Psi_{0,m}^{-1}\right]\cdot\Psi_{0,m} - p_{0,n}\left(I+2\overset{m}{E}_{m,n}(t)\right)^{-1}. \tag{26}$$

4. EQUATIONS OF MOTION (EQUILIBRIUM) AND BOUNDARY CONDITIONS

When the equations of motion (equilibrium) are derived in coordinates of different states for a particle existed in the "$n$"-th state, the equation in the "$n$"-th state is used. It has the usual form:

$$\overset{n}{\nabla}\cdot\overset{n}{\sigma}_{0,n} + \rho_n\overset{n}{F}_n = \rho_n\overset{n}{a}_n, \tag{27}$$

where $\rho_n$, $\overset{n}{F}_n$, $\overset{n}{a}_n$ are density, body force and acceleration of a particle in the "$n$"-th state, correspondingly.

Using (27) and identities

$$\rho_q = (1+\Delta_{q,p})\rho_p;\quad \overset{n}{\sum}_{0,n} = (1+\Delta_{0,n})\overset{n}{\sigma}_{0,n},\quad \overset{m}{e}{}^i = \overset{n}{e}{}^i\cdot\Psi_{m,n}^{\mathrm{T}}$$

(the last of these identities can be easily derived from (5)) and notation

$$\overset{m}{F}_n = \overset{n}{F}_n\cdot\Psi_{m,n}^{-1};\quad \overset{m}{a}_n = \overset{n}{a}_n\cdot\Psi_{m,n}^{-1},$$

we have, as shown by Levin (1988):

$$\overset{m}{\nabla}\cdot\overset{m}{\sum}_{0,n} - \overset{m}{\sum}_{0,n}\cdot\overset{m}{\nabla}\ln(1+\Delta_{0,n}) + \overset{m}{\sum}_{0,n}\cdot\cdot\overset{m}{\nabla}\Psi_{m,n}\cdot\Psi_{m,n}^{-1} -$$
$$-\left[\overset{m}{\nabla}\cdot\Psi_{m,n}^{-1}\cdot\Psi_{m,n}\right]\cdot\overset{m}{\sum}_{0,n} + \overset{m}{F}_n\rho_m(1+\Delta_{0,m}) = \overset{m}{a}_n\rho_m(1+\Delta_{0,m}). \tag{28}$$

We can obtain the boundary conditions by using the fact that, when equations of motion are derived, the boundary conditions for the total (for the "$m$"-th state) true stress tensor have a form:

$$\overset{m}{N}\cdot\overset{m}{\sigma}_{0,m} = \overset{m}{P}{}^{(m)}_{\overset{(m)}{N}} \tag{29}$$

where $\overset{(m)}{N}$ is the unit normal; $\overset{m}{P}{}^{(m)}_{\overset{(m)}{N}}$ is the true stress vector at the area element $\mathrm{d}\overset{m}{\tau}_k = \overset{(m)}{N}\,|\,\mathrm{d}\overset{m}{\tau}_k|$.

Note that subscript $k$ under $\tau$ is introduced only to distinguish notation for area element ($\mathrm{d}\tau_k$) from the notation for time ($\tau$).

Using the relation derived by Levin and Tarasiev (1980):

$$\overset{m}{\mathrm{d}\tau_k} = (1+\Delta_{q,m})\,\mathrm{d}\overset{q}{\tau_k}\cdot\Psi^{\mathrm{T}-1}_{q,m}, \tag{30}$$

we have:

$$\overset{q}{N}\cdot\overset{q}{\sum}_{0,m} = \overset{q}{P}{}^{(m)}_{\underset{N}{(q)}}, \tag{31}$$

where

$$\overset{q}{P}{}^{(m)}_{\underset{N}{(q)}} = (1+\Delta_{0,q})\frac{\left|\mathrm{d}\overset{q}{\tau_k}\right|}{\left|\mathrm{d}\overset{m}{\tau_k}\right|}\overset{m}{P}{}^{(m)}_{\underset{N}{(m)}}\cdot\Psi^{-1}_{m,q}. \tag{32}$$

Equation (30) can be developed. We start from the known equation presented, for example, by Eringen (1962) that may be written with the preceding notation as:

$$\mathrm{d}\overset{0}{\tau_k} = (1+\Delta_{0,1})^{-1}\,\mathrm{d}\overset{1}{\tau_k}\cdot\Psi^{\mathrm{T}}_{0,1}. \tag{33}$$

It is clear that in the case of some intermediate states one can write:

$$\mathrm{d}\overset{0}{\tau_k} = (1+\Delta_{0,q})^{-1}\,\mathrm{d}\overset{q}{\tau_k}\cdot\Psi^{\mathrm{T}}_{0,q},\quad d\overset{0}{\tau_k} = (1+\Delta_{0,m})^{-1}\,\mathrm{d}\overset{m}{\tau_k}\cdot\Psi^{\mathrm{T}}_{0,m}.$$

Then, eqn (30) follows from the identity

$$(1+\Delta_{0,q})^{-1}\,\mathrm{d}\overset{q}{\tau_k}\cdot\Psi^{\mathrm{T}}_{0,q} = (1+\Delta_{0,m})^{-1}\,\mathrm{d}\overset{m}{\tau_k}\cdot\Psi^{\mathrm{T}}_{0,m},$$

taking into account identities (6) and (18).

## 5. FORMULATION OF BOUNDARY-VALUE PROBLEMS OF REPEATEDLY SUPERIMPOSED LARGE DEFORMATIONS IN ELASTIC OR VISCOELASTIC BODIES

Equations of motion (28) combined with the boundary conditions (31), constitutive equations of the type of (23), (26) and kinematic relations in the form (1)–(10) constitute a closed system of equations in displacements for the case when stresses are prescribed at the boundary. The shape of the boundary can be characterized in spaces of different states, and change of boundary shape or boundary conditions (including step-wise change with large deformations in every step) in loading can be given additionally. Furthermore, change of boundary shape and boundary conditions can be defined as the repeated change of connectedness of a region, occupied by a body, in the process of loading. Indeed, let us consider the case when the body gains large initial deformations in transition from the initial to the first intermediate state. Then let us imagine a closed surface, following Levin (1988), Levin and Tarasiev (1980) and Levin and Bulatov (1983), and remove the part of a body bounded by this surface, and let the effect of this part of the remaining one be replaced on the principle of the releasing of braces by the forces, distributed over this surface. It is clear that this replacement doesn't change the state of stress and strain in the remaining part of a body. Then these forces passed to the category of external ones are quasistatically (for example, isothermally) varied by a large amount, for instance, reduced to zero, and a body passes into the second intermediate state, gaining (losing) large (at least, in the vicinity of newly formed boundary surface) additional deformations and stresses. The shape of the boundary surface is changed, too.

This procedure can be continued further, and for bodies of viscoelastic material, the moments new boundary surfaces are formed at are sufficient, as noted by Christensen (1971) and Ilyushin and Pobedrya (1970).

Note that the shape of the original (for example, at the first intermediate state) boundary surface in the problems of this type can be defined in the first intermediate state as well as in any subsequent state (for example, in the final state). In the second case, the problem is formulated and solved in the space of the final state where the shape of the boundary surface is given, but solution is extremely complex, because the system of equilibrium eqns (28) with boundary conditions (31) includes equlibrium equations and boundary conditions for tensor $\overset{N}{\sum}_{0,1}$, that depends not only on tensor $\overset{N}{E}_{0,1}$, but also on deformation gradient $\Psi_{1,N}$ or, in the same way, on displacement vectors $u_k$ $(k = 2,\dots N)$.

Note, additionally, that the relations obtained permit us to consider dynamic problems of the repeatedly superimposed large deformations and the stability problems, where loss of stability can take place in any stage of loading, that is, after the repeated change of boundaries and boundary conditions in the process of loading.

## 6. PLANE PROBLEM OF A CIRCULAR IN THE FINAL STATE HOLE

By the solution of problems in this paragraph the following constitutive relations will be used: for compressible materials—the Murnaghan potential introduced by Murnaghan (1951):

$$\overset{0}{\sum}_{0,n} = \overset{0}{a}_0\left(\overset{0}{E}_k\right)I + \overset{0}{a}_1\left(\overset{0}{E}_k\right)\overset{0}{E}_{0,n} + \overset{0}{a}_2\left(\overset{0}{E}_k\right)\left(\overset{0}{E}_{0,n}\right)^2,$$

where

$$\overset{0}{a}_0\left(\overset{0}{E}_k\right) = \lambda\overset{0}{E}_1 - 3C_3\left(\overset{0}{E}_1\right)^2 + C_4\overset{0}{E}_2;\quad \overset{0}{a}_1\left(\overset{0}{E}_k\right) = 2G + 2C_4\overset{0}{E}_1;\quad \overset{0}{a}_2\left(\overset{0}{E}_k\right) = 3C_5;$$

for incompressible materials—the Treloar potential considered by Treloar (1949)

$$\overset{n}{\sigma}_{0,n} = \mu F_{0,n} - p_n I,$$

where $F_{0,n} = \Psi^T_{0,n}\cdot\Psi_{0,n}$; at $n = 1$ $F_{0,1}$ is the Finger strain tensor; $p_n$ is a scalar determined from solution of equilibrium equation together with incompressibility conditions; $\mu$ is the shear modulus of small deformation theory.

Note that the basic qualitative difference between this problem and a similar problem in the pure linear statement (small deformations) is that in the linear problem the superposition principle is valid. And it means that this problem can be at once superseded with the standard inverse problem, that is, one can assume that a hole is not nucleated at the second step of deformation but exists initially in a body. Therefore, a need for consideration of step-by-step loading is loosen. The quantitative difference can be shown by consideration of specific cases.

Figure 1 shows the "true" total contour stress at different contour points ($\varphi = 0$; $\varphi = \pi/4$; $\varphi = \pi/2$; $\varphi$ is the polar angle of contour point) referred to the one in the linear case for Treloar material undergoing uniaxial initial loading ($\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}} = p$) as a function of initial load referred to the shear modulus ($p/\mu$). One can see, for example, that at $p/\mu = 0.7$ that corresponds to this case to initial deformations $\overset{0}{e}_{0,1_{11}} = -0.15$, $\overset{0}{e}_{0,1_{22}} = 0.22$, the correction for nonlinear effects is of the order of 35% (for stress).

Figure 2 shows the hole contour in the intermediate state at $p/\mu = 0.7$ for linear and nonlinear problems ($R$ is the radius of the contour in the final state). One can see that, in

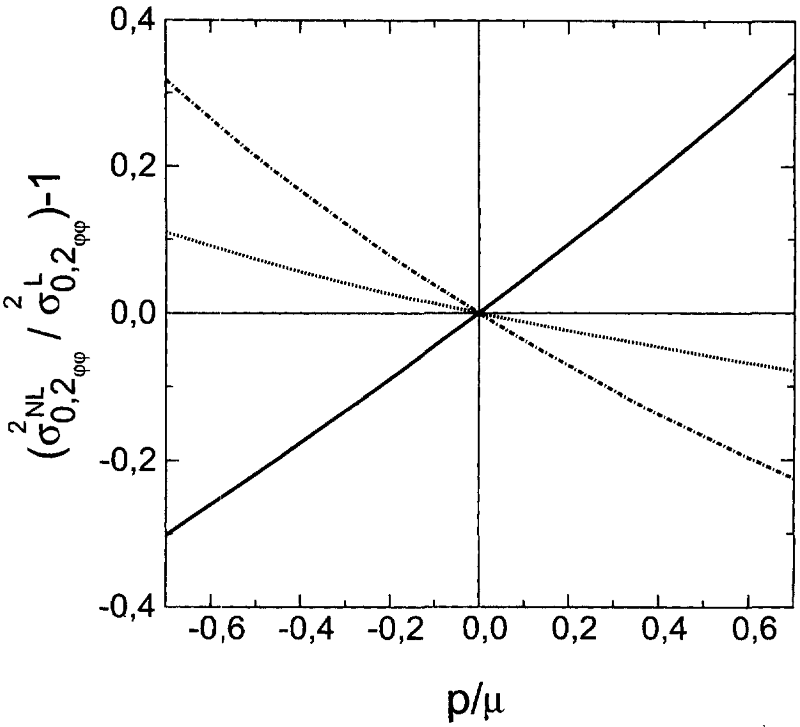


Fig. 1. Circular in the final state cavity. Treloar material. Uniaxial initial load: $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}} = p$. The correction for nonlinear effects for contour stress $(\overset{2}{\sigma}{}^{NL}_{0,2_{\varphi\varphi}}/\overset{2}{\sigma}{}^{L}_{0,2_{\varphi\varphi}} - 1)$ as a function of $p/\mu$ at different contour points ($\varphi$ is the polar coordinate of a contour point).

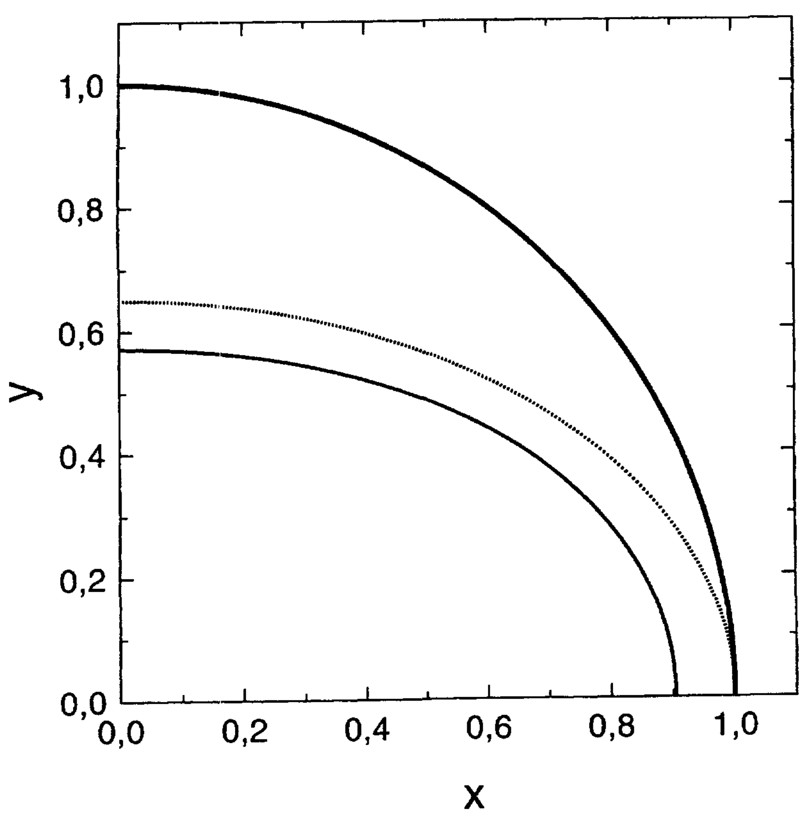


Fig. 2. Same problem as in Fig. 1. Cavity shape at the intermediate state at $p/\mu = 0.7$. Dotted line gives the linear approximation, as a reference.

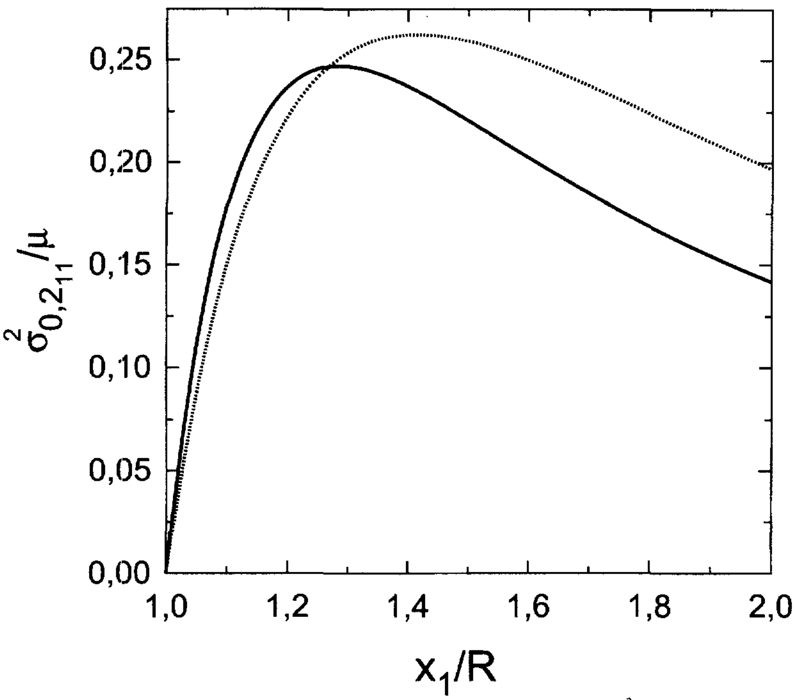


Fig. 3. Same problem as in Fig. 1. Variation of true stress component $\overset{2}{\sigma}_{0,2_{11}}$ referred to $\mu$ as a function of $x_1$ at $p/\mu = 0.7$. The solid line gives the nonlinear solution $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}/\mu$, and the dotted line gives the linear approximation $\overset{2}{\sigma}{}^{L}_{0,2_{11}}/\mu$, as a reference.

the first place, in linear case it is ellipse, and in nonlinear case it is not. In the second place, the correction for nonlinear effects is of the order of 22%.

Figure 3 shows the true total stress (Cauchy stress) tensor component $\overset{2}{\sigma}_{0,2_{11}}$ along $x_1$ axis for the same problem for nonlinear and linear cases (to tell one from the other, stress tensor component for the linear case has an additional index "$L$": $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$, and for the nonlinear case—the additional index "$NL$"). One can see that $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$ and $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ are at maximum at different points and a distance between these points is of the order of $0.1R$, and in the nonlinear case $\overset{2}{\sigma}_{0,2_{11}}$ reaches its maximum nearer to the contour.

Figure 4 shows the component $\overset{2}{\sigma}_{0,2_{11}}$ of the true total stress tensor (Cauchy stress) along the $x_1$ axis for the case $\overset{1}{\sigma}_{0,2_{22}} = -\overset{1}{\sigma}_{0,2_{11}} = p$, $p/\mu = -0.4$ (Treloar material). One can see that the distance between the points of minimum of $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$ and $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ is less than the correspondent distance on Fig. 3 and is of the order of $0.05R$, and the difference between the minimal values of $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$ and $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ is of the order of 21%. That is, the correction for nonlinear effects in this case is of the order of 21%, and, from another side, the points of minimum of $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$ and $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ are not the same.

Figure 5 shows the correction for nonlinear effects for the true total (Cauchy) hoop stress referred to this stress in the linear case $(\overset{2}{\sigma}{}^{NL}_{0,2_{\varphi\varphi}}/\overset{2}{\sigma}{}^{L}_{0,2_{\varphi\varphi}} - 1)$ for Murnaghan material $(\lambda/G = 2.24$; $C_3/G = -1.96$; $C_4/G = 3.61$; $C_5/G = -11.13$; these values of constants corresponds to copper, see Seeger and Buck (1960)) when $\overset{1}{E}_{0,2_{11}} = -\overset{1}{E}_{0,2_{22}} = e$ (strain shear) as a function of $e$ at $\varphi = 0$. One can see that the correction for nonlinear effects for stress is found to be 34% at $e = -0.04$.

Figure 6 shows $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$, $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ along the $x_1$ axis for the previous case when $e = -0.04$. One can see that the difference between the maximal values of $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$ and $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ is of the order of 28%.

Note that, in this paragraph and below, we mean by "linear solution" the solution for small deformations and linear constitutive equations.

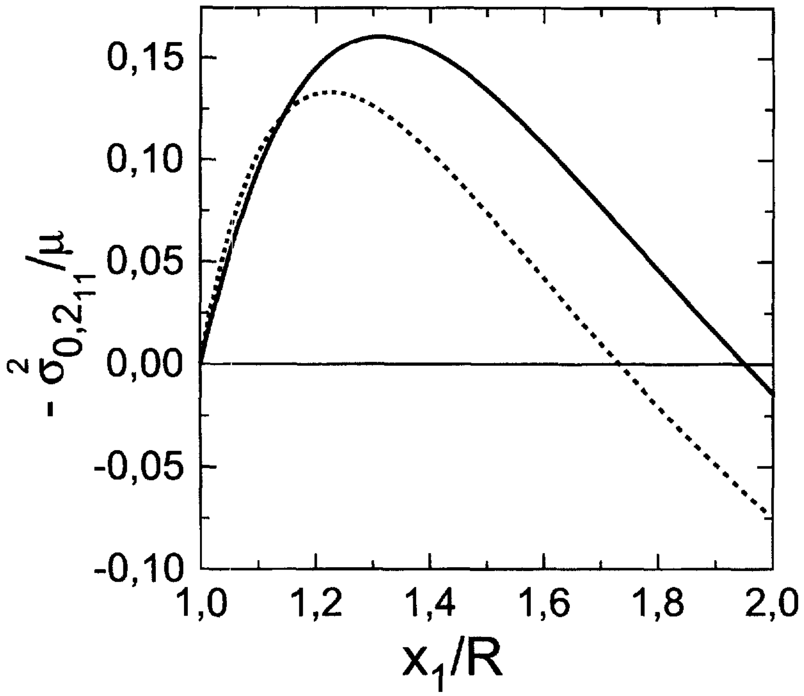


Fig. 4. Circular in the final state cavity. Treloar material. Initial load: stress shear; $\overset{1}{\sigma}_{0,1_{22}} = -\overset{1}{\sigma}_{0,1_{11}} = p$, $p/\mu = -0.4$. Variation of $\overset{2}{\sigma}_{0,2_{11}}$ referred to $\mu$ as a function of $x_1$. The solid line gives the nonlinear solution $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}/\mu$, and the dotted line gives the linear approximation $\sigma^{L}_{0,2_{11}}$, as a reference.

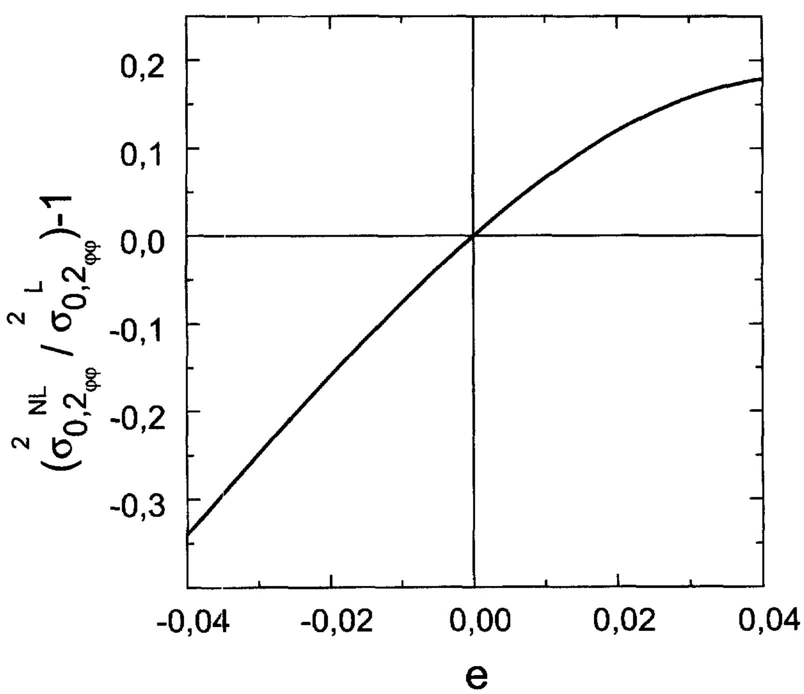


Fig. 5. Circular in the final state hole. Murnaghan material. $\lambda/G = 2.24$; $C_3/G = -1.96$; $C_4/G = 3.61$; $C_5/G = -11.13$. Initial deformation: strain shear; $E_{0,2_{11}} = -E_{0,2_{22}} = e$. The relative correction for nonlinear effects for contour stress $(\overset{2}{\sigma}{}^{NL}_{0,2_{\varphi\varphi}}/\overset{2}{\sigma}{}^{L}_{2_{\varphi\varphi}} - 1)$ as a function of $e$ at $\varphi = 0$.

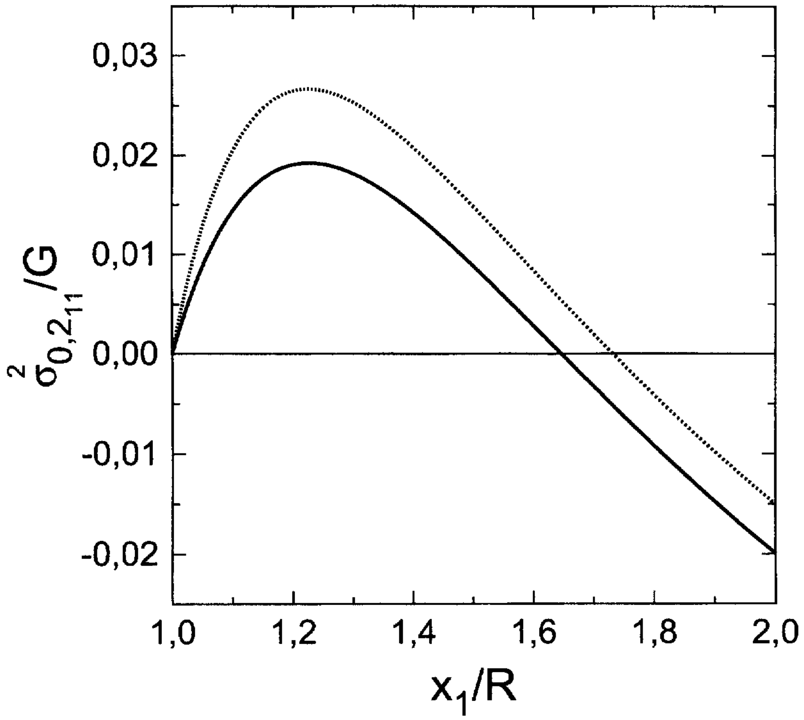


Fig. 6. Same problem as in Fig. 5. Variation of true stress component $\overset{2}{\sigma}_{0,2_{11}}$ referred to $G$ as a function of $x_1$ at $e = -0.04$. The solid line gives the nonlinear solution $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}/G$, and the dotted line gives the linear approximation $\overset{2}{\sigma}{}^{L}_{0,2_{11}}/G$, as a reference.

## 7. PLANE PROBLEM OF A CIRCULAR IN INTERMEDIATE STATE HOLE IN A BODY OF VISCOELASTIC MATERIAL

In this paragraph the problems for incompressible viscoelastic material, which mechanical properties are described by the constitutive relations (24), (25) with $l(t) = A\text{-}t^{\gamma-1}\exp(-at)$, where $\gamma = 0.016$, $aA^{1/(1-\gamma)} = 1.34 \cdot 10^{-4}$, will be considered. Note that the relaxation operators with singular kernels are used to describe the mechanical properties of viscoelastic materials by Bagley and Torvik (1986).

Statement of problem is the same as the one explained in Sections 5 and 6 for the case when the form of the surface is defined at the moment of formation. The problem is solved with the help of "SUPERPOSITION" program package. We set that the initial load is applied at $t = 0$, and hole is nucleated at $t = t_0$.

Figure 7 shows the correction for nonlinear effects for additional displacement component $u_{2_2}$ of some contour points ($\varphi = \pi/4$; $\varphi = \pi/2$) referred to this component for the linear case $-(u^{NL}_{2_2}/u^{L}_{2_2} - 1)$ as a function of $t$ at $t_0 = 0.0744A^{1/(1-\gamma)}$, $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}} = p$, $p/\mu_0 = 0.1$ (a logarithmic scale for $t$ is used). One can see that with rise to $t$ the correction for nonlinear effect decreases.

Figure 8 shows $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}$, $\overset{2}{\sigma}{}^{L}_{0,2_{11}}$ along the $x_1$ axis for the case $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}} = p$, $p/\mu_0 = -0.1$, $t_0 = 0.0744A^{1/(1-\gamma)}$, $t/t_0 = 100$. One can see that the distance between the points of maximum of $|\overset{2}{\sigma}{}^{NL}_{0,2_{11}}|$ and $|\overset{2}{\sigma}{}^{L}_{0,2_{11}}|$ is over $0.05R$, and

$$\frac{\max\left|\overset{2}{\sigma}{}^{NL}_{0,2_{11}}\right|}{\max\left|\overset{2}{\sigma}{}^{L}_{0,2_{11}}\right|} = 0.93.$$

Note that for this case

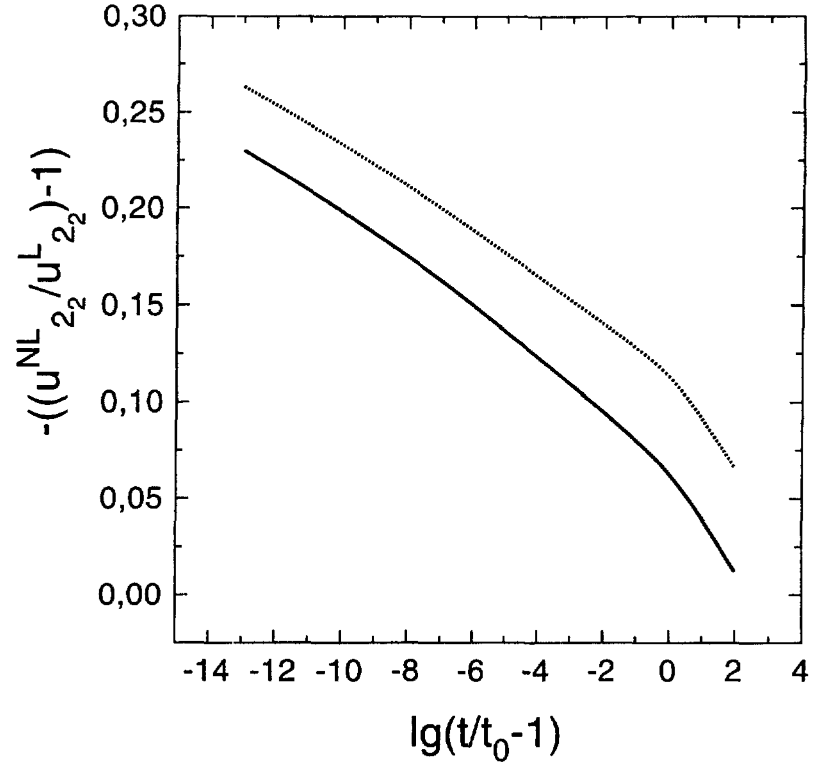


Fig. 7. Circular in the intermediate state hole. Viscoelastic material. $l(t) = At^{\gamma-1}\exp(-at)$; $\gamma = 0.016$, $aA^{1/(1-\gamma)} = 1.34 \cdot 10^{-4}$; $t_0 = 0.0744A^{1/(1-\gamma)}$. Uniaxial initial load: $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}} = p$, $p/\mu_0 = 0.1$. The relative correction for nonlinear effects for additional displacement component $u_{2_2}$ of some contour points $(u^{NL}_{2_2}/u^{L}_{2_2} - 1)$ as a function of $lg(t/t_0 - 1)$ at different contour points (the solid line corresponds to $\varphi = \pi/4$, and the dotted line corresponds to $\varphi = \pi/2$).

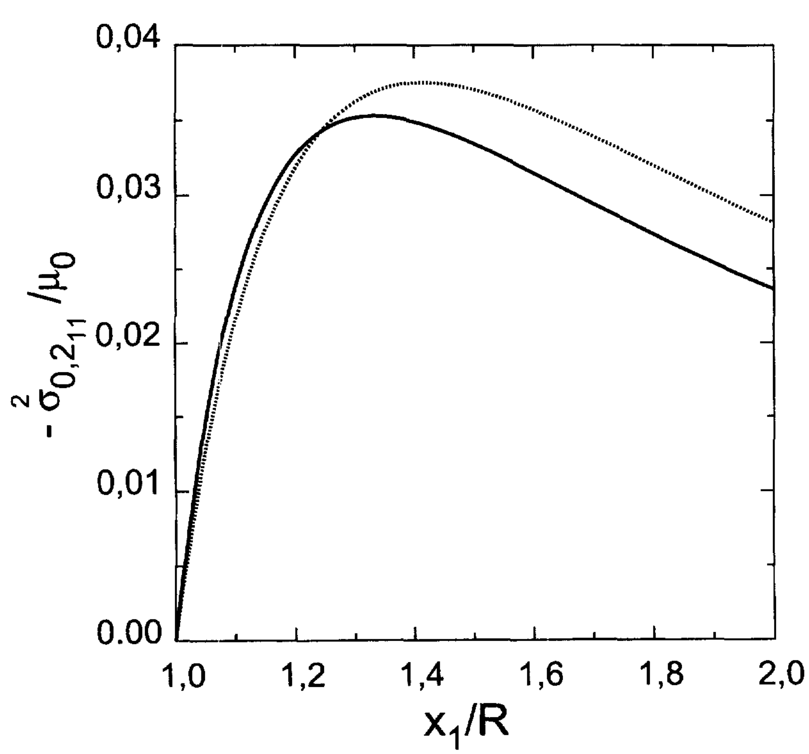


Fig. 8. Same problem as in Fig. 7. Variation of true stress component $\overset{2}{\sigma}_{0,2_{11}}$ referred to $\mu_0$ as a function of $x_1$ at $p/\mu_0 = -0.1$, $t/t_0 = 100$. The solid line gives the nonlinear solution $\overset{2}{\sigma}{}^{NL}_{0,2_{11}}/\mu_0$, and the dotted line gives the linear approximation $\overset{2}{\sigma}{}^{L}_{0,2_{11}}/\mu_0$, as a reference.

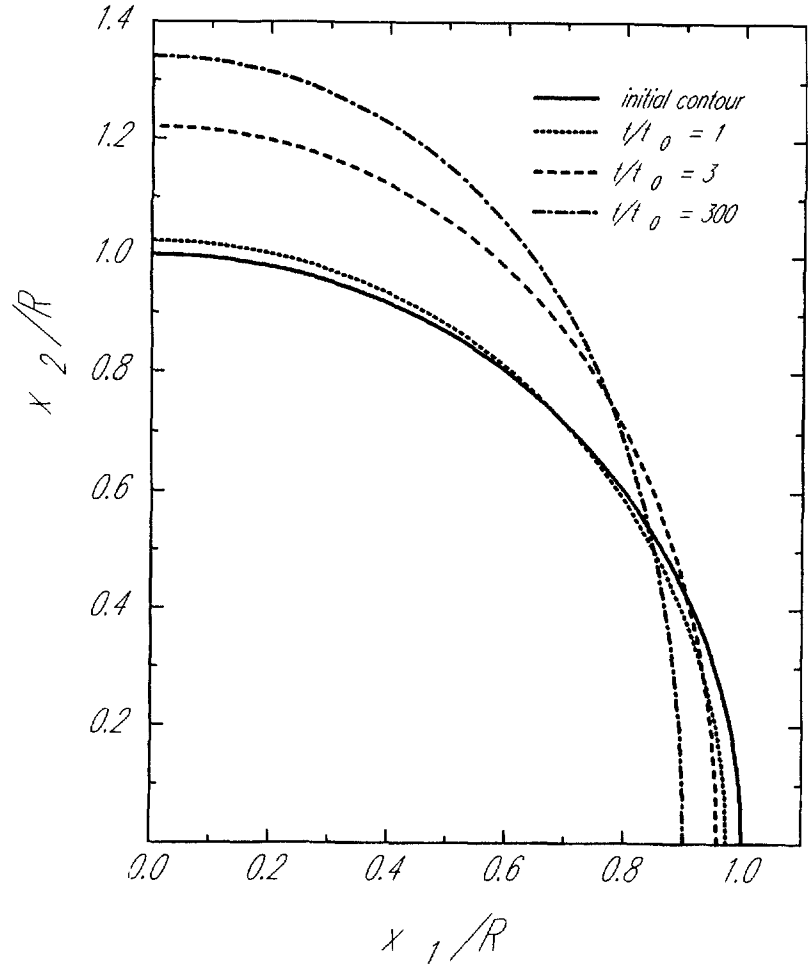


Fig. 9. Same problem as in Fig. 7. Contour shape at different moments of time. The dotted line corresponds to $t/t_0 = 1$, the dashed line corresponds to $t/t_0 = 3$, and the dot-and-dash line corresponds to $t/t_0 = 300$. The circular solid line corresponds to a quarter of the initial hole contour at the moment of nucleation.

$$\frac{\max\left|\overset{2}{\sigma}{}^{NL}_{0,2_{22}}\right|}{\max\left|\overset{2}{\sigma}{}^{L}_{0,2_{22}}\right|} = 1.21.$$

And, finally, Fig. 9 shows the contour of the boundary surface at different moments of time $t$ when $p/\mu_0 = 0.1$, $t_0 = 0.0744A^{1/(1-\gamma)}$. One can see that, unlike the linear case, it is not elliptic.

## 8. DISCUSSION

The formulated theory of repeatedly superimposed large deformations in elastic or viscoelastic materials allows one to state and to solve problems in which boundary conditions are changed repeatedly including a change of connectedness of a region occupied by a body. This theory permits to solve a new class of problems on stress concentration, when the new stress concentrators are nucleated in the process of loading in bodies with large deformations producing new additional strains superimposed on the initial ones. Results of solution of specific problems (Sections 6 and 7) show, particularly, that the corrections for nonlinear effects are as great as 25–35% ( for the given parameters of loading and material properties). For example, for the Treloar material in the problem on nucleation of a hole assuming a circular shape at the final state (Section 6) in a body (with large initial deformations) at the final state a correction for nonlinear effects is of the order of 35% for the initial stress $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,1_{22}}/\mu = 0.7$. It is clear that for the elliptical holes the value of correction at the points of maximal stress concentration will be depending on the value of shape ratio $m$. For example, for the problem on the elliptical at the moment of formation hole in a body consisted of Treloar material with large initial deformations the correction referred to the maximal contour stress is of the order of 17% at $m = 0.6$, and it is of the order of 33% at $m = 0.8$ (for the initial stress $\overset{1}{\sigma}_{0,1_{11}} = 0$, $\overset{1}{\sigma}_{0,2_{22}}/\mu = 0.25$).

It should be noted that depending on type and parameters of material and hole shape a correction for nonlinear effects may be sufficient at a moderately large initial strain. For example, one can see from Figs 1 and 2 that the correction for nonlinear effects for maximal hoop stress is of the order of 35%, and the one for displacement is of the order of 22%. As is obvious from Fig. 5, at $\overset{1}{E}_{0,2_{11}} = -0.04$, $\overset{1}{E}_{0,2_{22}} = 0.04$ the correction for nonlinear effects for hoop stress is of the order of 34%. In the author's opinion, it is partially caused by stress concentration effects when large additional strains and stresses exceeding the initial ones and superimposed on them appear in a body due to nucleation of a stress concentrator.

In general, the boundary shape may be defined at an arbitrary state, and, consequently, three kinds of problems that may be solved with the use of theory of repeatedly superimposed large deformations may be separated: when the boundary shape is defined at the initial state, at any intermediate state and at the final state.

The theory constructed additionally permits one to consider new types of stability problems at finite deformations, when loss of stability may happen at any stage of loading depending on the sequence of application of external loads. This theory may be also supplied to the dynamic problems, for example, to the problem of the impact of shock waves on the massif of elastic or viscoelastic material with vertical or horizontal working, with varied temperatures field as with constant ones.

## 9. CONCLUSION

The theory of repeatedly superimposed large deformations in elastic or viscoelastic materials is presented. On the basis of this theory, a statement of boundary-value problems is formulated for the case of stepwise loading of a bodies. This statement permits one to solve problems with repeated change of boundaries and boundary conditions in loading. This permits us specifically to solve (at large deformations) a new class of problems on stress concentration when new stress concentrators are nucleated in a body with large initial deformations producing large (at least, in the neighborhood of a new stress concentrator) deformations. New stress concentrators may be specifically formed by removal or addition of parts to the previously loaded body.

The particular model problems of nucleation of circular and elliptical cavities in a loaded body are considered. Cases when the cavity shape is given at the moment of nucleation, cases when this shape is known at the final state, or (for viscoelastic state) at the given moment of time, are investigated. The last kind of problem also requires determination of the shape of nucleating cavity at the moment of formation.

Results of solution of problems for the specific cases show that allowing for finiteness of strains gives a correction of the order of 25–35% at the point of maximal stress concentration in comparison with the solution of the same problem based on the assumption of small deformations superimposed on the large ones.

In subsequent work it is planned to present the results of solutions to problems of the formation of some holes permitting specifically the optimization of stress fields using interaction of holes (by change of their shape, sequence of formation and location).

In conclusion, the author notes that many thoughts and ideas of Prof. G. S. Tarasiev are developed in this work.

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